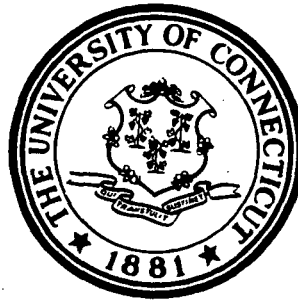# The University of Connecticut
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AN ADAPTIVE SCHEME FOR OBSERVING THE
STATE OF AN UNKNOWN LINEAR SYSTEM

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Summary

A full order adaptive observer is described for observing the states of a single-input single-output observable continuous differential system with unknown parameters. Convergence of the observer states to those of the system is accomplished by directly changing the parameters of the observer using an adaptive law based upon Liapunov stability theory. Observer eigenvalues may be freely chosen. Some restriction is placed upon the system input in that it must be sufficiently rich in frequencies in order to insure convergence.

Introduction

The Luenberger observer [1,2,3] allows extraction of all the states of an observable linear system when given the output and the parameters of the system. In some cases the system parameters may not be known; consequently, in these cases the state observations are subject to error. Previous investigators of this phenomenon have attempted to estimate the error [4] or change the observer parameters in some beneficial way [5]. Their analysis suffers in that the error cannot be guaranteed to vanish. We report a full order observer for a restrictive class of systems (that is, single-input single-output observable continuous linear differential systems in the absence of a deterministic or random disturbance vector) for which the observation error is guaranteed to vanish regardless of the size of the constant or slowly varying parameter ignorance. The observer parameters are directly changed in a way that satisfies a quadratic Liapunov function of the error and the correct but unknown Luenberger observer parameters. The observer poles may be placed at any stable location and no derivatives are required in the adaptive law.

The Problem

A differential system is assumed of the form

$$\dot{w} = Aw + Br \qquad w(0) = w^0$$
$$y = [1\ 0\ 0\ \text{---}\ 0]w \qquad (1)$$

A nxn
B nx1

for which only the single output $y = Cw = w_1$ is available for measurement. It is assumed that

some or all of the elements of matrices $\bar{A}$ and B are unknown, $\bar{A}$ is stable, w^0 may be unknown, and the pair $(C,\bar{A})$ is completely observable. The observer is of the form

$$\dot{z} = Fz + GCw + Dr + Hu \qquad z(0) = z^0 \qquad (2)$$

F nxn   G nx1
D nx1   H nxn and diagonal

where z is arbitrary and u is a control vector yet to be defined but with the property that $u \to 0$ as $t \to \infty$. The problem is to adaptively form a triple (G, D_1, T) so that the error vector defined as $e = z - T^{-1}w$ vanishes as the system adapts. T is a non-singular square matrix with the property that $CT = C$.

Define a transformation $x = T^{-1}w$ so that $e = z - x$. Then (1) becomes

$$\dot{x} = \bar{A}_0 x + T^{-1}Br \qquad x(0) = Tw^0$$
$$y = CTx = Cx \qquad (1A)$$
$$\bar{A}_0 = T^{-1}\bar{A}T$$

and (2) becomes

$$\dot{z} = Fz + GCx + Dr + Hu$$
$$z(0) = z^0 \qquad (2A)$$

It is desired that $\bar{A}_0 = T^{-1}\bar{A}T$ be in the "output" form

$$\bar{A}_0 = \begin{bmatrix} -a_{11} & 1 & 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & 1 & 0 & \cdots & 0 \\ -a_{31} & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ -a_{n1} & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

wherein the first column contains the system parameters and all other elements are zero save the super diagonal elements, which are unity. It is clear that for any non-zero matrix $\bar{A}$ there corresponds a similar matrix $\bar{A}_0$, although the elements of the similarity transformation may be unknown if elements of $\bar{A}$ are unknown. The following theorem defines the additional restriction that must be placed upon $\bar{A}$ so that both $\bar{A}_0 = T^{-1}\bar{A}T$ and

$CT = C$.

Theorem [proof given in ref. 11]. Let A be an nxn matrix, $C = [k,0,0,\ldots,0]$ a 1xn matrix with $k \neq 0$, A_0 an nxn matrix in output form, and $T = [\;.\;\begin{smallmatrix} C \\ \hat{T} \end{smallmatrix}\;.\;]$ an nxn nonsingular matrix. There exists an (n-1)xn matrix $\hat{T}$ such that $A = TA_0 T^{-1}$ iff the pair (C,A) is completely observable. As a result of the theorem, any observable system (1) may be placed by similarity transformation into system (1A) with $CT = C$. The elements of T may be unknown since $\hat{A}$ is unknown. The problem will be considered as defined by equation (1A) and (2A), so that $e = z-x$ must vanish. Eventually the problem of constructing w from x will be solved.

The Adaptive Law

It is now assumed without restriction that some stable "nominal" plant matrix is known or is chosen so that $\tilde{A}_0 = A_0 + \Delta A_0$ where A_0 has all known elements and is in output form. Then ΔA_0 contains all zero elements except for the first column which has unknown elements. The vector error equation may then be written, letting $e = z-x$, as

$$\dot{e} = Fe + (F + GC - A_0 - \Delta A_0)x + \Delta Br + Hu$$

where $\Delta B = D - T^{-1}B$. F may be chosen as $F = A_0 - GC$. The resulting error equation is

$$\dot{e} = (A_0 - GC)e - \Delta A_0 x + \Delta Br + Hu \tag{3}$$

It is desired to reduce ΔA_0, ΔB, and u to zero. Then, if G is chosen so that $A_0 - GC$ is stable, $e(t)$ approaches zero. A theorem of Luenberger [1] allows the eigenvalues of $A_0 - GC$ to be placed arbitrarily by selection of G, with the sole exception that $A_0 - GC$ cannot have the same eigenvalues as A_0. Consequently, $A_0 - GC$ can always be made stable. Moreover, it is assumed throughout that G will be chosen so that all elements of $A_0 - GC$ will be constant even under changes in ΔA_0 due to the adaptive law.

The error between plant states x and observer states z may be measured only by the scalar $e_1 = z_1 - y = z_1 - x_1$. To insure that only available measurements are included in the adaptive law, the vector error equation (3) is "collapsed" to yield a scalar differential equation of the form

$$\sum_{i=0}^{n} k_i e_1^{(i)} = \sum_{i=0}^{n-1} \alpha_i x_1^{(i)}$$

$$+ \sum_{i=0}^{m} \beta_i r^{(i)} + \sum_{i=0}^{n-1} h_i u_i^{(i)} \tag{4}$$

where

$k_i \in K_0 = A_0 - GC$, a constant matrix

$\alpha_i \in -\Delta A_0$ and its several derivatives

$\beta_i \in \Delta B$ and its several derivatives

$h_i \in H$

Letting $p = d/dt$, the left side of (4) may be written as

$$\prod_{i=1}^{n} (p + \lambda_i) e_1$$

Now a reduction of order technique, similar to that of Gilbart and Monopoli [6], will be applied. n-1 of the $(p + \lambda_i)$ terms will be selected and factored out of the right side of (4) excluding the u_i terms. Assuming that $p + \lambda_1$ was not selected, the error equation then has the form

$$(p + \lambda_1) \prod_{i=2}^{n} (p+\lambda_i) e_1 = \prod_{i=2}^{n} (p+\lambda_i)\left[\sum_{i=0}^{n+m+1} \phi_i v_i\right]$$

$$- f(\phi_j^{(i)}, v_\ell^{(k)}) + \sum_{i=0}^{n-1} h_i u_i^{(i)} \tag{5}$$

where

$$\prod_{j=2}^{n} (p + \lambda_j)v_i = x_1^{(i)} \qquad i=1,2,\ldots,n-1$$

$$\prod_{j=2}^{n} (p + \lambda_j)v_i = r^{(i-n)} \qquad i=n,n+1,\ldots,n+m$$

$$v_0 = x_1$$

ϕ_i are functions of $\{\alpha_i\}$ and $\{\beta_i\}$, and $f(\phi_j^{(i)}, v_\ell^{(k)})$ is a function of derivatives of ϕ_j but does not contain ϕ_j for any j.

Then associated with each $\phi_j^{(i+1)}$ is a $u_i^{(i)}$; specifically, $u_i^{(i)}$ is made equal to the negative of all the terms in which $\phi_j^{(i+1)}$ appears for every i and j. Here it is noted that by construction neither v_i nor u_i require a derivative network for implementation. Then (5) becomes

$$(p+\lambda_1) \prod_{i=2}^{n} (p+\lambda_i) e_1 = \prod_{i=2}^{n} (p+\lambda_i)\left[\sum_{i=0}^{n+m+1} \phi_i v_i\right]$$

$$\tag{6}$$

and each $u_i = g(\phi_j^{(1)}, v_\ell^{(k)})$

Taking Laplace transform of (6) and dividing by $\prod_{i=2}^{n}(s + \lambda_i)$ yields

$$(s+\lambda_1)e_1 = L\left[\sum_{i=0}^{n+m+1}\phi_i v_i\right]$$

$$+(\text{initial conditions})/\prod_{i=2}^{n}(s+\lambda_i) \tag{7}$$

for which follows

$$\dot{e}_1 + \lambda_1 e_1 = \sum_{i=0}^{n+m+1}\phi_i v_i + \sum_{i=0}^{n}\psi_i \exp[-\lambda_i t],$$

$$\lambda_1 \text{ real}$$

where ψ_i are constants depending upon the initial conditions if $\{\lambda_i\}$ are distinct; otherwise some ψ_i may be time dependent. (Note: should it be desired, when n is even, to have no real observer pole, the operation in (7) may still be made by modifying the right side of (7) in an obvious way).

A Liapunov function $V = \frac{1}{2}(m_0 e_1^2 + \sum_{i=1}^{n+m+1}m_i \phi_i^2)$

is chosen, and $\dot{V}$ is calculated. Following Shackcloth [7], $\dot{V}$ can be made to be of the form

$$\dot{V} = -m_0\lambda_1 e_1^2 + e_1\sum_{i=1}^{n}\psi_i \exp[-\lambda_i t] \tag{9}$$

when $\quad \dot{\phi}_i = -\frac{m_0}{m_i}v_i e_1 \quad$ for all i $\tag{10}$

Other adaptive laws can easily be chosen instead if it is desired to increase convergence speed [8,9].

From the form of $\dot{V}$, e_1 is stable in the sense of Lagrange with the region of attraction determined by the unknown constants ψ_i and the exponential time function. Clearly the region of attraction shrinks exponentially with time and eventually vanishes; consequently e_1 is eventually asymptotically stable, and $\lim_{t\to\infty} e_1 = 0$. All derivatives of e_1 must vanish in the limit also since the error equation is linear of first order. Although the Liapunov function is defined on a non-compact manifold (i.e., $\dot{V}$ contains e_1 but not ϕ_i), it can be shown that $\{\phi_i\}$ is eventually asymptotically stable if the input to the plant, r, is periodic and contains $(n+m+1)/2$ distinct frequencies, none of which has a phase shift of $k\pi$ through the plant, k any integer. [10] It must be assumed that the adaptive observer is limited to systems for which convergence to parameter differences $\{\phi_i\}$ is assured.

Since $\Delta A_0, \Delta B_0, u_i$ approach zero, the vector e is eventually asymptotically stable if G is chosen so that A_0-GC is asymptotically stable.

Reconstruction of T

Using the "nominal" matrix as initial conditions, the actual value of the system parameters may be determined by integrating the change in parameters $\{\phi_i\}$ until adaptation is complete and combining appropriately. This procedure may be accomplished while the adaptation progresses by forming a matrix $\tilde{T}(t)$ with elements composed of the combination of nominal values and the integrals $\int_0^t \phi_i dt$ where $\{\dot{\phi}_i\}$ is defined in (10).

$\hat{\omega}$, the estimate of ω, is constructed from the observer output z by forming $\tilde{T}$ z. Since $\lim_{t\to\infty} z = x$ and $\lim_{t\to\infty}\tilde{T} = T$, so $\lim \hat{\omega} = \omega$.

Example

A third order plant with one zero is considered for illustration of the previous discussed design. Let the plant be described by

$$\dot{w} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(a_0+\alpha_0) & -(a_1+\alpha_1) & -(a_2+\alpha_2) \end{bmatrix} w + \begin{bmatrix} 0 \\ c_1 \\ c_2 \end{bmatrix} r$$

$$y = w_1 \tag{1*}$$

where $\alpha_0, \alpha_1, \alpha_2$, C_1, and C_0 are unknown. A transformation T that delivers the system into output form is

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a_2+\alpha_2 & 1 & 0 \\ a_1+\alpha_1 & a_2+\alpha_2 & 1 \end{bmatrix}$$

Note that C = CT. Then in output form, (1*) becomes

$$\dot{x} = \begin{bmatrix} -(a_2+\alpha_2) & 1 & 0 \\ -(a_1+\alpha_1) & 0 & 1 \\ -(a_0+\alpha_0) & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ b_1+\beta_1 \\ b_0+\beta_0 \end{bmatrix} r$$

$$y = x_1 = w_1 \tag{1A*}$$

The error equation is

$$\dot{e} = \begin{bmatrix} -(a_2+g_2) & 1 & 0 \\ -(a_1+g_1) & 0 & 1 \\ -(a_0+g_0) & 0 & 0 \end{bmatrix} e + \begin{bmatrix} \alpha_2 & 0 & 0 \\ \alpha_1 & 0 & 0 \\ \alpha_0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -\beta_1 \\ -\beta_0 \end{bmatrix} r$$

$$+ \begin{bmatrix} 0 \\ u_1 \\ u_2 \end{bmatrix} \tag{3*}$$

and the scalar error equation is

$$\dddot{e}_1 + k_2\ddot{e}_1 + k_1\dot{e}_1 + k_0e_1 = (\alpha_0 + \dot\alpha_1 + \ddot\alpha_2)x_1 + (\alpha_1 + 2\dot\alpha_2)\dot{x}_1$$

$$+ \alpha_2\ddot{x}_1 - \beta_1\dot{r} - (\beta_0 + \dot\beta_1)r + \dot{u}_1 + u_0 \qquad (4*)$$

where

$$k_0 = a_0 + g_0$$
$$k_1 = a_1 + g_1$$
$$k_2 = a_2 + g_2$$

Let $s^3 + k_2 s^2 + k_1 s + k_0 = (s+\lambda_1)(s+\lambda_2)(s+\lambda_3)$. Then

$$(p + \lambda_1)(p + \lambda_2)(p + \lambda_3)e_1 =$$

$$(p+\lambda_2)(p+\lambda_3)[\phi_0 v_0 + \phi_1 v_1 + \phi_2 v_2 - \phi_3 v_3 - \phi_4 v_4]$$

$$+ \dot\phi_2 v_0 - \dot\phi_3 r - \dot\phi_2 \dot{v}_2 - (\lambda_2+\lambda_3)\dot\phi_2 v_2$$

$$- \frac{d}{dt}(\dot\phi_2 v_2) - \dot\phi_1 \dot{v}_1 - (\lambda_2+\lambda_3)\dot\phi_1 v_1$$

$$- \frac{d}{dt}(\dot\phi_1 v_1) + (\lambda_2+\lambda_3)\dot\phi_3 v_3 + \dot\phi_3 \dot{v}_3$$

$$+ \frac{d}{dt}(\dot\phi_3 v_3) + (\lambda_2+\lambda_3)\dot\phi_4 v_4 + \dot\phi_4 \dot{v}_4$$

$$+ \frac{d}{dt}(\dot\phi_4 v_4) + \dot{u}_1 + u_0 \qquad (5*)$$

where

$$\phi_0 = \alpha_2 \qquad\qquad v_0 = x_1$$
$$\phi_1 = \alpha_0 - \lambda_2\lambda_3\alpha_2 \qquad \dot{v}_1 + (\lambda_2+\lambda_3)\dot{v}_1 + \lambda_2\lambda_3 v_1 = x_1$$
$$\phi_2 = \alpha_1 - (\lambda_2+\lambda_3)\alpha_2 \qquad \ddot{v}_2 + (\lambda_2+\lambda_3)\dot{v}_2 + \lambda_2\lambda_3 v_2 = \dot{x}_1$$
$$\phi_3 = \beta_1 \qquad\qquad \ddot{v}_3 + (\lambda_2+\lambda_3)\dot{v}_3 + \lambda_2\lambda_3 v_3 = \dot{r}$$
$$\phi_4 = \beta_0 \qquad\qquad \ddot{v}_4 + (\lambda_2+\lambda_3)\dot{v}_4 + \lambda_2\lambda_3 v_4 = r$$

Let

$$u_1 = - \dot\phi_4 v_4 - \dot\phi_3 v_4 + \dot\phi_1 v_1 + \dot\phi_2 v_2$$

$$u_0 = \dot\phi_2[-v_0 + \dot{v}_2 + (\lambda_2+\lambda_3)v_2] + \dot\phi_1[\dot{v}_1 + (\lambda_2+\lambda_3)v_1]$$

$$- \dot\phi_3[r + \dot{v}_3 + (\lambda_2+\lambda_3)v_3] - \dot\phi_4[\dot{v}_4 + (\lambda_2+\lambda_3)v_4]$$

Then

$$(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)e_1 =$$

$$(s+\lambda_2)(s+\lambda_3)\left[\sum_{i=0}^{2}\phi_i v_i - \sum_{i=3}^{4}\phi_i v_i\right]$$

$$+ \sum_{i=0}^{2} n_i s^i \qquad (7*)$$

Where n_i are unknown constants depending upon initial conditions. Dividing by $(s+\lambda_2)(s+\lambda_3)$ yields

$$\dot{e}_1 + \lambda_1 e_1 = \sum_{i=0}^{2}\phi_i v_i - \sum_{i=3}^{4}\phi_i v_i$$

$$+ \psi_1 \exp[-\lambda_2 t] + \psi_2 \exp[-\lambda_3 t) \qquad (8*)$$

Consequently,

$$\dot\phi_i = - \left(\frac{m_0}{m_{i+1}}\right) v_i e_1 \qquad i = 0,1,2$$

$$\dot\phi_i = \left(\frac{m_0}{m_{i+1}}\right) v_i e_1 \qquad i = 3,4 \qquad (10*)$$

The observer has the form

$$\dot{z} = \begin{bmatrix} -k_2 & 1 & 0 \\ -k_1 & 0 & 1 \\ -k_0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} g_2 & 0 & 0 \\ g_1 & 0 & 0 \\ g_0 & 0 & 0 \end{bmatrix} w_1$$

$$+ \begin{bmatrix} 0 \\ b_1 \\ b_0 \end{bmatrix} r + \begin{bmatrix} 0 \\ u_1 \\ u_0 \end{bmatrix}$$

where

$$\dot{g}_2 = - \frac{m_0}{m_3} e_1 v_0$$

$$\dot{g}_1 = - \left(\frac{m_0}{m_3} v_2 + (\lambda_2+\lambda_3)\frac{m_0}{m_1} v_0\right)e_1$$

$$\dot{g}_0 = - \left(\frac{m_0}{m_2} v_1 + \lambda_2\lambda_3 \frac{m_0}{m_1} v_0\right)e_1$$

$$\dot{b}_1 = - \frac{m_0}{m_4} e_1 v_3$$

$$\dot{b}_0 = - \frac{m_0}{m_5} e_1 v_4$$

and

$$\hat{\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ a_2 - \int_0^t \dot\alpha_2 dt & 1 & 0 \\ & 0 & \\ a_1 - \int_0^t \dot\alpha_1 dt & a_2 - \int_0^t \dot\alpha_2 dt & 1 \end{bmatrix}^{-1} z \qquad (*)$$

A Simulation

The third order system of the example was simulated on a digital computer using the following parameters

$$a_0 = 24 \quad \alpha_0 = 0 \quad C_1 = 30 \quad k_0 = 24 \quad m_0/m_3 = 8000$$

$$a_1 = 26 \quad \alpha_1 = 74 \quad C_2 = 195 \quad k_1 = 26 \quad m_0/m_S = 2000$$

$$a_2 = 9 \quad \alpha_2 = 0 \quad b_1 = 30 \quad k_2 = 9 \quad g_0 = g_2 = 0$$

The eigenvalues of the observer (determined by $\{k_i\}$) were $\lambda_1 = -4$, $\lambda_2 = -2$, $\lambda_3 = -3$. The input to the plant was a square wave of magnitude 1 and frequency 6t. Two parameters, b_0 and g_1, were adjusted by the adaptive law. These were initially set at $b_0 = 73$, $g_1 = -5$ corresponding to a correct value of $b_0 = 75$, $g_1 = -74$. The accompanying graph illustrates the behavior of b_0, g_1, e_2, and e_3 as a function of time.

Remark

As has been previously stated, $\hat{\omega} = \hat{T}z$ and $\lim_{t \to \infty} \hat{\omega} = \omega$. In the general case of an arbitrary plant matrix A, the determinant of $\hat{T}$ may vanish for some instances of time. These momentary occurrences, of course, have no detrimental effect on $\hat{\omega}$ since convergence of $\hat{\omega}$ to ω is guaranteed. In the important particular case of the preceding example, however, advantage has been taken of the fact that det $\hat{T}$ is constant by writing equation (*) as $\hat{\omega} = (\hat{T}^{-1})^{-1} z$. Since for the case of phase variable plant of high order the literal form of T^{-1} is easily produced, it is surmised that writing $(\hat{T}^{-1})^{-1} = \hat{T}$ allows a particularly simple construction of $\hat{\omega}$ when digital computation, rather than analog, is desired.

Conclusion

An adaptive observer has been demonstrated for single-input single-output systems with constant or slowly varying parameters. Work is currently underway to extend the observer to multivariable systems as well as systems with rapidly varying parameters. It is hoped that the adaptive ovserver will be eventually used not only for observing the state of an unknown system but in model reference problems and pole placement problems as well.

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b_0 vs. t
$-g_1$ vs. t

e vs. t